with K. Mkrtchyan [1401.xxxx & 14??.xxxx]

SFT 2014

On Higher-Spin Algebras

Euihun JOUNG

APC, Université Paris 7

What is **Higher-Spin Algebra (HSA)** ?

- Higher-spin extension of $\{ P_a \, , \, M_{ab} \}$
  - Gauging $\{ P_a \, , \, M_{ab} \}$ ➜ Gravity 
  - Gauging **HSA** ➜ Higher-spin Gauge Theory
  - Q) but, must we have **HSA** ?
    A) Yes, because $\cdots$

  - Q) so, what is **HSA**?
    A) inf. dim. Lie algebra such that $\cdots$

HS Gauge Theory given by $S[g_{\mu\nu}, \varphi_{\mu\nu\rho\sigma}, \dots]$

Gauge Sym

$$\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)} + \mathcal{O}(\varphi)$$

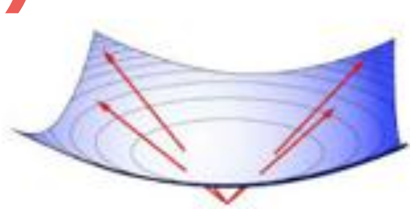
$$\delta \varphi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\varepsilon_{\mu_2\cdots\mu_s)} + \mathcal{O}(\varphi)$$

Background

$$\bar g_{\mu\nu} \,,\; \bar\varphi_{\mu_1\cdots\mu_s}$$

(A)dS metric

0



Global Sym

$$\bar\nabla_{(\mu}\bar\xi_{\nu)} + \cancel{\mathcal{O}(\varphi)} = 0$$

$$\bar\nabla_{(\mu_1}\bar\varepsilon_{\mu_2\cdots\mu_s)} + \cancel{\mathcal{O}(\varphi)} = 0$$

Killing Eq. $\bar{\nabla}_{(\mu_1}\bar{\varepsilon}_{\mu_2\cdots\mu_s)} = 0$

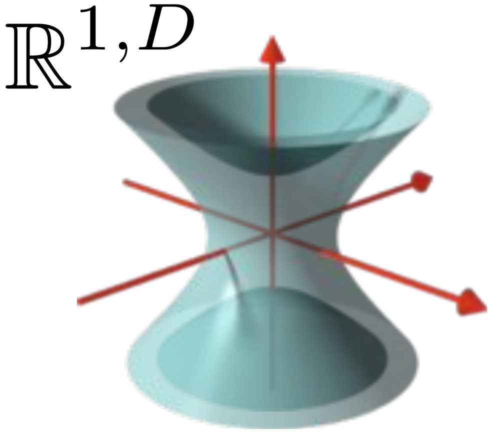
ex. flat bg

$\partial_{(\mu}\bar{\xi}_{\nu)} = 0$

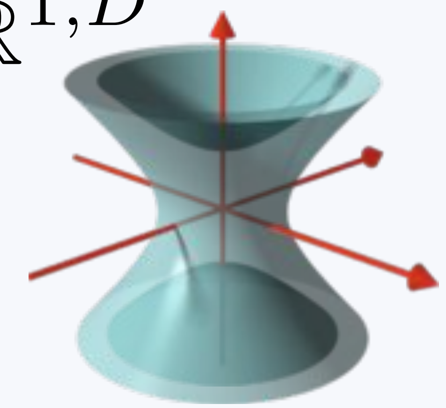
$$\bar{\xi}_\mu = v_a\,(P^a)_\mu + w_{ab}\,(M^{ab})_\mu$$

with $(P^a)_\mu = \delta^a_\mu$ and $(M^{ab})_\mu = x^{[a}\,\delta^{b]}_\mu$

※ Ambient-Space formulation



※ Ambient-Space formulation



(A)dS tensor field — Amb-Space tensor field

$$f_{\mu_1\cdots\mu_r}(x) \quad \longleftrightarrow \quad F_{M_1\cdots M_r}(X)$$

with $\boxed{(X\cdot\partial_X-\Delta_h)F=0=X\cdot\partial_U\,F}$

$$F(X,U)=\frac{1}{r!}\,F_{M_1\cdots M_r}(X)\,U^{M_1}\dots U^{M_r}$$

Gauge field & parameter

$$\varphi_{\mu_1\cdots\mu_s}(x)\ ,\ \varepsilon_{\mu_1\cdots\mu_r}(x) \quad \longleftrightarrow \quad \Phi(X,U)\ ,\ E(X,U)$$

with

$$(X\cdot\partial_X-U\cdot\partial_U+2)\,\Phi=0=X\cdot\partial_U\Phi$$
$$(X\cdot\partial_X-U\cdot\partial_U)\,E=0=X\cdot\partial_U E$$

then

$$\boxed{\delta\varphi_{\mu_1\cdots\mu_s}=\bar{\nabla}_{(\mu_1}\,\varepsilon_{\mu_2\cdots\mu_s)} \quad \longleftrightarrow \quad \delta\,\Phi=U\cdot\partial_X\,E}$$

traceless condition

Killing Eq. $\bar{\nabla}_{(\mu_1}\bar{\varepsilon}_{\mu_2\cdots\mu_s)} = 0$ $\eta^{\mu_1\mu_2}\,\bar{\varepsilon}_{\mu_1\cdots\mu_r} = 0$

ex. flat bg

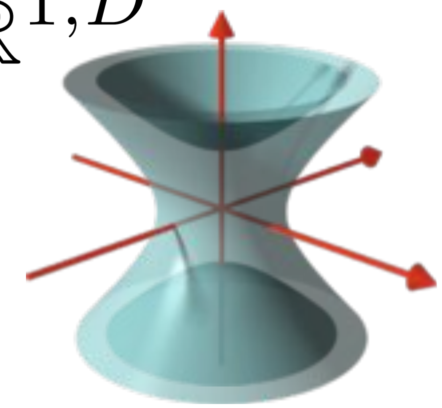
$\partial_{(\mu}\bar{\xi}_{\nu)} = 0$

$\bar{\xi}_\mu = v_a\,(P^a)_\mu + w_{ab}\,(M^{ab})_\mu$

with $(P^a)_\mu = \delta^a_\mu$ and $(M^{ab})_\mu = x^{[a}\,\delta^{b]}_\mu$

※ Ambient-Space formulation

$\mathbb{R}^{1,D}$



$\mathfrak{sp}_2$

$U\cdot\partial_X\,\bar{E} = 0$ $\partial_U^2\,\bar{E} = 0$

$X\cdot\partial_U\,\bar{E} = 0$ $(X\cdot\partial_X - U\cdot\partial_U)\,\bar{E} = 0$

traceless traceless

$\bar{E} = W_{A_1B_1,\dots A_rB_r}\,M^{A_1B_1,\dots A_rB_r}$

with $M^{A_1B_1,\dots A_rB_r}(X,U) = X^{[A_1}U^{B_1]}\dots X^{[A_r}U^{B_r]}$

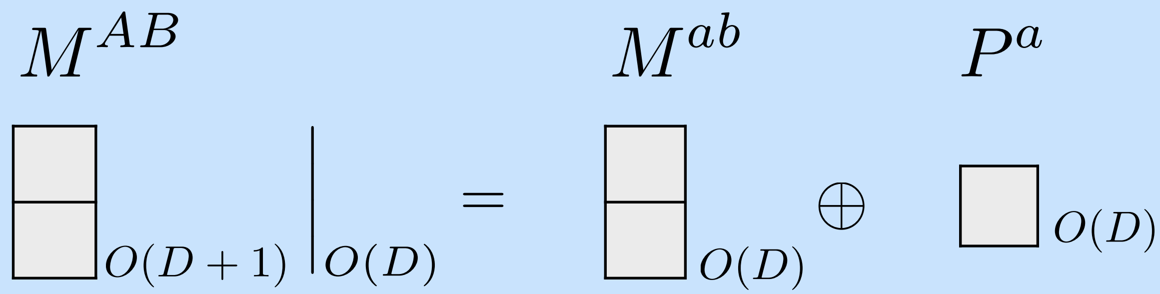
– traces

Higher-Spin Lie Algebra

- Vector space

$$\mathrm{Span}\{\, M^{A_1 B_1, \dots, A_r B_r} \,\}$$

branching ex.



- Lie bracket
  - Covariant HS charges:

$$\left[\, M^{A_1 B_1, \dots, A_r B_r} \,,\, M^{CD} \,\right] = \eta^{A_1 C}\, M^{D B_1, \dots, A_r B_r} + \cdots$$

  - Other brackets?

Action with Gauge Sym

Consider a theory with **fields** φ_i and an **action** $S[\varphi_i]$

deWitt notation

- Gauge sym $\delta_\varepsilon \varphi_i$ $\qquad$ $\delta_\varepsilon S[\varphi_i] = 0$

Consequences

- **Closure** symmetry

$$\left[\, \delta_{\varepsilon_1} \,,\, \delta_{\varepsilon_2} \,\right] = \delta_{[\varepsilon_2,\varepsilon_1]} + C_{ij}(\varepsilon_1,\varepsilon_2)\,\frac{\delta S}{\delta\varphi_i}\,\frac{\delta}{\delta\varphi_j}$$

- **Jacobi identity** transformation

$$\sum_{\text{cyclic}} \left[\, \delta_{\varepsilon_1} \,,\, \left[\, \delta_{\varepsilon_2} \,,\, \delta_{\varepsilon_3} \,\right] \right] = 0$$

Expanding in φ_i and Considering
bg $\bar{\varphi}_i = 0$ and global sym $\bar{\varepsilon}$ s.t. $\delta^{(0)}_{\bar{\varepsilon}} = 0$

$$\delta^{(0)}_{[\bar{\varepsilon}_1, \bar{\varepsilon}_2]^{(0)}} = 0$$

- Global sym is closed under $[\ ,\ ]^{(0)}$
  Lie bracket

Admissibility condition Konstein, Vasiliev

$$\delta^{(1)}_{\bar{\varepsilon}}\, S^{(2)} = 0 \qquad\qquad \left[\, \delta^{(1)}_{\bar{\varepsilon}_1} \,,\, \delta^{(1)}_{\bar{\varepsilon}_2} \,\right] S^{(2)} = \delta^{(1)}_{[\bar{\varepsilon}_2, \bar{\varepsilon}_1]^{(0)}}\, S^{(2)}$$

- $\delta^{(1)}_{\bar{\varepsilon}}$: Rep. of global sym **Lie algebra**

with Jacobi identity

HSA (and its Rep) and Cubic Interactions

Consistent **Lie brackets** of HSA can be

- extracted from cubic interactions   EJ, Taronna '13

  Brink, Bengtsson, Metsaev, Bekaert, Boulanger,
  Fotopoulos, Tsulaia, Manvelyan, Mkrtchyan, Ruehl, …

- used to construct **non-Abelian** cubic interactions

  Fradkin,Vasiliev, Skvortsov, Boulanger, Ponomarev, …

- Lie brackets $\Leftrightarrow$ non-Abelian (NA) cubic couplings
- Jacobi id. $\Leftrightarrow$ Coupling Cnst of NA couplings
- Represent. $\Leftrightarrow$ Coupling Cnst of Current couplings

What is this Lie algebra?

- Oscillator-based construction

  - 4D $\quad [\, y_\alpha \overset{\star}{,} y_\beta \,] = \epsilon_{\alpha\beta} \qquad [\, \bar{y}_{\dot\alpha} \overset{\star}{,} \bar{y}_{\dot\beta} \,] = \epsilon_{\dot\alpha\dot\beta} \quad \alpha,\beta = 1,2$

    Fradkin, Vasiliev

  - 3D $\quad [\, y_\alpha \overset{\star}{,} y_\beta \,] = \epsilon_{\alpha\beta}(1 + \nu\, k) \quad \alpha,\beta = 1,2$

    arbi. para.

    Bergshoeff, Blencowe, Stelle; Vasiliev; Pope, Romans, Shen; …

  - 5D $\quad [\, y_\alpha \overset{\star}{,} \bar{y}^\beta \,] = \delta_\alpha^\beta \qquad y_\alpha\, \bar{y}^\alpha \sim 0 \quad \alpha,\beta = 1,2,3,4$

    Fradkin, Linetsky; Sezgin, Sundell; Vasiliev

  - Any D $\quad [\, Y_\alpha^a \overset{\star}{,} Y_\beta^b \,] = \eta^{ab}\, \epsilon_{\alpha\beta} \qquad \eta_{ab}\, Y_\alpha^a\, Y_\beta^b \sim 0$

    $\alpha,\beta = 1,2 \qquad a,b = 0,1,\ldots,D$

    Vasiliev

- Construction from UEA of isometry algebra

  HSA = UEA / Ideal Iazeolla, Sundell

> “ Determine the **least number** of **DoF** for which
> a **QM system** admits a given semisimple **Lie algebra**
> and construct the corresponding class of realizations ”
>
> abstract “Minimal Realizations and Spectrum Generating Algebras” — Joseph ’74

Minimum number of **oscillator pairs** necessary to represent a given semisimple **Lie algebra**

Minimal Representation:

- Irrep with the minimum Gelfand-Kirillov (GK) dim

# of continuous variables necessary
to describe the rep. vector space

	minimum GK dim
$\mathfrak{sp}_{2N}$	N
$\mathfrak{sl}_N$	$N-1$
$\mathfrak{so}_N$	$N-3$

Minimal Representation:

- Irrep with the minimum Gelfand-Kirillov (GK) dim

# of continuous variables necessary
to describe the rep. vector space

Remarks

- GK dim of a D-dim "particle": $\quad D-1$
- GK dim of MR of $\mathfrak{so}_{D+1}$ (iso of AdS_D): $\quad D-2$
- Flato-Fronsal: $2\times(D-2)=(D-1)+(D-3)$
- HSA $\mathfrak{g}$ of D-dim theory with a finite spectrum

$$\mathfrak{g}\supset\mathfrak{so}_{D+1} \qquad\qquad D-2\leq\pi(\mathfrak{g})\leq D-1$$

GK dim of MR

Minimal Representation:

- Irrep with the minimum Gelfand-Kirillov (GK) dim

  # of continuous variables necessary to describe the rep. vector space

	minimum GK dim
$\mathfrak{sp}_{2N}$	N
$\mathfrak{sl}_N$	$N-1$
$\mathfrak{so}_N$	$N-3$

- Unique except for $\mathfrak{sl}_N$ cf. Boulanger, Ponomarev, Skvortsov, Taronna
- Associated with the minimal (nilpotent coadj.) orbit
- Joseph ideal: annihilator of MR in UEA

Classical Lie algebras (notation)

$\mathfrak{sp}_{2N}$	$[\![\, N_{AB}\,,\ N_{CD}\,]\!] = \Omega_{A(C}\, N_{D)B} + \Omega_{B(C}\, N_{D)A}$
$\mathfrak{sl}_N$	$[\![\, L^a{}_b\,,\ L^c{}_d\,]\!] = \delta^c_b\, L^a{}_d - \delta^a_d\, L^c{}_b$
$\mathfrak{so}_N$	$[\![\, M_{ab}\,, M_{cd}\,]\!] = 2\,(\eta_{a[c}\, M_{d]b} - \eta_{b[c}\, M_{d]a})$

$$N_{AB} = N_{BA} \qquad \Omega_{AB} = -\Omega_{BA} \qquad\qquad M_{ab} = -M_{ba}$$

$$A, B, \ldots = 1, \ldots, 2N \qquad\qquad a, b, \ldots = 1, \ldots, N$$

Minimal (nilpotent coadjoint) orbit

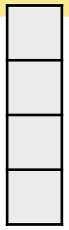
$$N(U) = \frac{1}{2}\, N_{AB}\, U^{AB}\,, \qquad L(V) = L^a{}_b\, V_a{}^b\,, \qquad M(W) = \frac{1}{2}\, M_{ab}\, W^{ab}$$

$$\begin{aligned}
\mathcal{O}_{\min}(\mathfrak{sp}_{2N}) &: \quad U^{A[B}\, U^{D]C} = 0\,,\\
\mathcal{O}_{\min}(\mathfrak{sl}_{N}) &: \quad V_{[a}{}^b\, V_{c]}{}^d = 0\,,\\
\mathcal{O}_{\min}(\mathfrak{so}_{N}) &: \quad W^{a[b}\, W^{cd]} = 0 = W^{ab}\, W_b{}^c\,,
\end{aligned}$$

Joseph ideal

Joseph; … ; Fronsdal; …

$$\begin{aligned}
\mathcal{J}(\mathfrak{sp}_{2N}) &: \quad N_{A[B} \otimes N_{C]D} + \frac{\hbar}{2}\left(\Omega_{A[B}\, N_{C]D} + \Omega_{D[B}\, N_{C]A} - \Omega_{BC}\, N_{AD}\right)\\
&\qquad\qquad + \frac{\hbar^2}{2}\left(\Omega_{A[B}\, \Omega_{C]D} - \Omega_{BC}\, \Omega_{AD}\right) \sim 0\,,\\
\mathcal{J}(\mathfrak{sl}_{N}) &: \quad L^{[a}{}_b \otimes L^{c]}{}_d + \hbar\left(\delta^{[a}_{(\beta}\, L^{c]}{}_{d)} + \lambda\, \delta^{[a}_{[b}\, L^{c]}{}_{d]}\right) + \hbar^2\, \frac{\lambda^2 - 1}{4}\, \delta^{[a}_{[b}\, \delta^{c]}_{d]} \sim 0\,,\\
\mathcal{J}(\mathfrak{so}_{N}) &: \quad M_{a[b} \otimes M_{cd]} - \hbar\, \eta_{a[b}\, M_{cd]} \sim 0 \sim M_{c(a} \otimes M_{b)}{}^c - \hbar^2\, \frac{N-4}{2}\, \eta_{ab}\,,
\end{aligned}$$



Minimal Representation

$\mathfrak{sp}_{2N}$: quadratic in **N** oscillator(-pair)s (metaplectic rep)

$\mathfrak{sl}_N$: linear, quadratic and cubic in **N-1** oscillators

$\mathfrak{so}_N$: non-polynomial in **N-3** oscillators Gunaydin, et al

Reductive Dual Pair Correspondence (Howe duality)

$$(\,G_1\,,\,G_2\,)\subset Sp_{2N}$$

1-dim rep $\pi_1 \longleftrightarrow \pi_2$

$$\left[\,y_{\alpha a}\stackrel{\star}{,}y_{\beta b}\,\right]=\epsilon_{\alpha\beta}\,\eta_{ab}$$

$$\alpha,\beta=\pm\qquad a,b=1,\dots,N$$

$$(\,G_1\,,\,G_2\,)=\underbrace{\boxed{\overset{\mathfrak{sl}_N}{(\,GL_1\,,\,GL_N\,)}}}_{y_{+a}\,y_-{}^a\,\sim\,\lambda}\quad\text{and}\quad\underbrace{\boxed{\overset{\mathfrak{so}_N}{(\,Sp_2\,,\,O_N\,)}}}_{y_{\alpha a}\,y_\beta{}^a\,\sim\,0}$$

Some Generalizations

$$hs = \mathcal{U} \Big/ \left(\ydiagram{2} \oplus \ydiagram{1,1,1,1} \right)$$

A $\mathcal{I}(\nu) = \left(\ydiagram{1,1,1,1} \oplus (C_2 - \nu) \right)$

B $\mathcal{J}(\nu) = \left(\ydiagram{2} \oplus (C_2 - \nu) \right)$

Boulanger, Skvortsov

A Partially-massless related

- **ideal** for $\nu_\ell = -\dfrac{(D-1-2\,\ell)(D-1+2\,\ell)}{4}$

- **coset algebra** $\mathfrak{p}_\ell \simeq \bigoplus_{p=0}^{\ell-1} \bigoplus_{r=2p}^{\infty} \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{r} \\ \hline r-2p & 2p \\ \hline \end{array}$

  Partially-massless HSA

  Eastwood, Leistner; Gover, Silhan; Michel; Bekaert, Grigoriev

- directly obtainable with $\mathcal{I}_\ell = \left(\begin{array}{|c|}\hline 2\ell \\ \hline \end{array} \oplus \ydiagram{1,1,1,1} \right)$

- correspond to **ℓ—dim rep** of Sp_2

Some Generalizations

$$hs = \mathcal{U} \,\Big/ \left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \right) \quad \Longrightarrow$$

A $\mathcal{I}(\nu) = \left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \oplus (C_2 - \nu) \right)$

B $\mathcal{J}(\nu) = \left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus (C_2 - \nu) \right)$

Boulanger, Skvortsov

B Mixed-symmetry related

even D

- quotienting out $\begin{array}{|c|}\hline \ \\ \hline D \\ \hline \ \\ \hline\end{array}$

$$(\nu - \nu_2) \cdots (\nu - \nu_{D/2}) = 0 \qquad \nu_k = \frac{D+1}{2}(k-1)\left(k - \frac{D+1}{2}\right)$$

scalar spinor

odd D

- split into two copies
- ideal for $\nu(h) = \frac{D+1}{2}(h-1)\left(h + \frac{D-3}{2}\right)$ helicity-h singleton rep

Vasiliev; Bekaert, Grigoriev

- directly obtainable with

$$\mathcal{J}_h = \left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|}\hline 2h-1 \\ \hline \ \\ \hline\end{array} \right), \qquad \mathcal{J}'_h = \left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|}\hline 2h+1 \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \right)$$


Thank you