Dual Pair Correspondence in Physics

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Collaboration with

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based on 2006.07102

Dual Pair Correspondence

- Oscillator Realizations of Lie groups
  - "Good and Old"
  - Importance in Physics and Mathematics

- Spinor-helicity formulation
- Twistor theory
- Unfolded formulation

- Reductive dual pair correspondence (Howe duality)
  - Roger Howe (1976, published in 1989)
  - Theta correspondence

Our original motivations

- Spinor-helicity formulation
  - Massive and (A)dS generalizations
  - Other dimensions
- Higher spin business
  - Any *d* (partially-massless) higher spin algebras
  - Relevant representations (minimal representations etc)

Oscillator realization

- **Jordan-Schwinger** map (1935)
  - *SU*(2)

$$J_3 = \frac{1}{2}(a^\dagger a - b^\dagger b)\,, \qquad J_+ = a^\dagger\, b\,, \qquad J_- = b^\dagger\, a\,,$$

  - *U*(1) (number operator)

$$N = a^\dagger\, a + b^\dagger\, b\,, \qquad N\,|\Psi_j\rangle = 2j\,|\Psi_j\rangle\,.$$

  - 1-to-1 correspondence
    - *SU*(2) irrep j ↔ *U*(1) irrep 2j

Dual Pair Correspondence

- N set of oscillators ➔ Heisenberg Group H_N ➔ $Sp(2N,\mathbf{R})$
- Metaplectic representation $\mathcal{W}$ (**Weil-Segal-Shale**)
  - Double cover of $Sp(2N,\mathbf{R})$
  - $Sp(2,\mathbf{R}) \cong SU(1,1) \cong SL(2,\mathbf{R})$: double cover of $SO(1,2)$

Dual Pair Correspondence

- Reductive dual pair (*G* , *G'*)
  - Reductive subgroups in $Sp(2N,\mathbf{R})$
  - Mutual centralizers [*g* , *g'*]=0
- Restriction of $\mathcal{W}$ to *G* x *G'* : $$\mathcal{W}\,|_{G\times G'} = \bigoplus_{\zeta\in\Sigma^G_{\mathcal{W}}} \pi_G(\zeta)\otimes\pi_{G'}\big(\theta(\zeta)\big)$$
  - $\mathrm{mult}_{\mathcal{W}}(\pi_G(\zeta)\big) = \dim\ \pi_{G'}\big(\theta(\zeta)\big)$

    $\mathrm{mult}_{\mathcal{W}}\Big(\pi_{G'}\big(\theta(\zeta)\big)\Big) = \dim\ \pi_G(\zeta)$

$$\begin{array}{cccc} \theta\ : & \Sigma^G_{\mathcal{W}} & \longrightarrow & \Sigma^{G'}_{\mathcal{W}} \\ & \zeta & \longmapsto & \theta(\zeta) \end{array}$$

Historical digression 1/3

- **Wigner** (1937): ($U(4)$,$U(N)$), **Racah** (1943): ($Sp(N)$,$Sp(M)$)
- Nuclear physics: Interacting boson model, Nuclear shell model, etc
- Dynamical group, Spectrum generating group, etc
  - Review: **Rowe, Carvalho and Repka,** *"Dual pairing of symmetry groups and dynamical groups in physics"*, 1207.0148
- Coherent state physics (1970s~)

Historical digression 2/3

- **Oscillator representations of non-compact Lie groups**
  - $SO(1,d)$: **Dirac** (1944), “expansor”
  - $SO(1,3)$: **Harish-Chandra** (1947), “expinor”
  - SO(2,3): **Dirac** (1963), “Remarkable representation”
  - $SO(2,4)$: **Kursunoglu** (1962), **Mack-Todorov** (1969), …
  - $SU(2,2|N)$, $OSp(N|4,\mathbf{R})$, $OSp(8^*|N)$: **Gunaydin et al** (1980s)

Historical digression 3/3

- **Higher spin (HS) gravity**
  - Spinorial AdS_4 : **(Fradkin-)Vasiliev** (1988-1992)
  - Spinorial AdS_5 & AdS_7 : **Sezgin-Sundell** (2001 & 2002)
  - Vectorial AdS_{d+1} : **Vasiliev** (2003)
  - Many related works : **Alkalaev, Bekaert, Boulanger, Mkrtchyan, Grigoriev, Iazeolla, Skvortsov, Sundell, …**
- **Dual pair correspondences (in Mathematics)**
  - Explicit analysis of classical Lie groups in 1990s~

Irreducible Dual Pairs

- Any dual pair (G, G') has the form,

$$G = G_1 \times G_2 \times \cdots \times G_p\,, \qquad G' = G'_1 \times G'_2 \times \cdots \times G'_p$$

where (G_i, G'_i) is a real form of (GL_M, GL_N) or (O_N, Sp_{2M})

Embedding group	(G, G')
$Sp(2MN, \mathbb{R})$	$\big(GL(M,\mathbb{R}), GL(N,\mathbb{R})\big)$
$Sp(4MN, \mathbb{R})$	$\big(GL(M,\mathbb{C}), GL(N,\mathbb{C})\big)$
$Sp(8MN, \mathbb{R})$	$\big(U^*(2M), U^*(2N)\big)$
$Sp(2(M_+ + M_-)(N_+ + N_-), \mathbb{R})$	$\big(U(M_+, M_-), U(N_+, N_-)\big)$
$Sp(2M(N_+ + N_-), \mathbb{R})$	$\big(O(N_+, N_-), Sp(2M, \mathbb{R})\big)$
$Sp(4MN, \mathbb{R})$	$\big(O(N,\mathbb{C}), Sp(2M,\mathbb{C})\big)$
$Sp(4N(M_+ + M_-), \mathbb{R})$	$\big(O^*(2N), Sp(M_+, M_-)\big)$

Irreducible Dual Pairs

- (GL_M, GL_N)

$$(\omega^I_A\,,\ \tilde{\omega}^A_I)\,, \qquad A = 1,\dots,M\,, \quad I = 1,\dots,N\,,$$

$$[\omega^I_A, \tilde{\omega}^B_J] = \delta^B_A\,\delta^I_J\,, \qquad [\omega^I_A\,, \omega^J_B] = 0 = [\tilde{\omega}^A_I\,, \tilde{\omega}^B_J]\,.$$

$$X^A{}_B = \frac{1}{2}\{\tilde{\omega}^A_I, \omega^I_B\} = \tilde{\omega}^A_I\,\omega^I_B + \frac{N}{2}\,\delta^A_B\,, \qquad R_I{}^J = \frac{1}{2}\{\tilde{\omega}^A_I, \omega^J_A\} = \tilde{\omega}^A_I\,\omega^J_A + \frac{M}{2}\,\delta^J_I$$

- (O_N , Sp_{2M})

$$y^I_A\,, \qquad A = 1,\dots,N\,, \quad I = 1,\dots,2M\,,$$

$$[y^I_A, y^J_B] = E_{AB}\,\Omega^{IJ}\,,$$

$$M_{AB} = \Omega_{IJ}\,y^I_{[A}\,y^J_{B]}\,, \qquad K^{IJ} = E^{AB}\,y^{(I}_A\,y^{J)}_B$$

Oscillator (Fock) realization

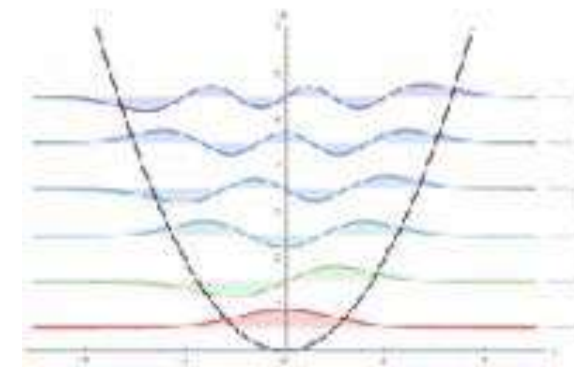
	$\big(GL(M,\mathbb{R}),GL(N,\mathbb{R})\big) \subset Sp(2MN,\mathbb{R})$	$\big(GL(M,\mathbb{C}),GL(N,\mathbb{C})\big) \subset Sp(4MN,\mathbb{R})$	$\big(U^*(2M),U^*(2N)\big) \subset Sp(8MN,\mathbb{R})$	$\big(U(m,n),U(p,q)\big) \subset Sp(2MN,\mathbb{R})$
$\dagger$	$(\omega^I_A)^\dagger = \omega^I_A, \quad (\tilde\omega^A_I)^\dagger = -\tilde\omega^A_I$ $A=1,\dots,M, \quad I=1,\dots,N$	$(\omega^I_A)^\dagger = \omega^I_A{}^*, \quad (\tilde\omega^A_I)^\dagger = -\tilde\omega^{A*}_I$ $A=1,\dots,M, \quad I=1,\dots,N$	$(\omega^I_A)^\dagger = \Omega_{IJ}\,\Omega^{AB}\,\omega^J_B, \quad (\tilde\omega^A_I)^\dagger = -\Omega^{IJ}\,\Omega_{AB}\,\tilde\omega^B_J$ $A=1,\dots,2M, \quad I=1,\dots,2N$	$(\omega^I_A)^\dagger = \eta_{AB}\,\eta^{IJ}\,\tilde\omega^B_J$ $A=(a,\mathtt{a}),\ a=1,\dots,m,\ \mathtt{a}=1,\dots,n$ $I=(i,\mathtt{i}),\ i=1,\dots,p,\ \mathtt{i}=1,\dots,q$
$(\alpha,\tilde\alpha)$	$a^I_A := \frac{1}{\sqrt2}\left(\omega^I_A - \tilde\omega^A_I\right)$	$a^I_A := \frac{1}{\sqrt2}\left(\omega^I_A - \tilde\omega^{A*}_I\right), \quad b^I_A := \frac{1}{\sqrt2}\left(\omega^I_A{}^* - \tilde\omega^A_I\right)$	$a^I_A := \frac{1}{\sqrt2}\left(\omega^I_A - \Omega_{AB}\,\Omega^{IJ}\,\tilde\omega^B_J\right)$	$a^i_a := \omega^i_a, \quad b^{\mathtt{a}}_i := \tilde\omega^{\mathtt{a}}_i, \quad c^a_{\mathtt{i}} := \tilde\omega^a_{\mathtt{i}}, \quad d^{\mathtt{i}}_{\mathtt{a}} := \omega^{\mathtt{i}}_{\mathtt{a}}$
$\mathfrak{g}$	$X^A{}_B = \frac12\left(\tilde a^A_I\,a^I_B - \tilde a^B_I\,a^I_A + \tilde a^A_I\,\tilde a^B_I - a^I_A\,a^I_B\right)$	$X^A_{\pm B} = X^A{}_B \pm (X^A{}_B)^*$ $X^A{}_B = \frac12\left(\tilde a^A_I\,a^I_B - \tilde b^B_I\,b^I_A + \tilde a^A_I\,\tilde b^B_I - b^I_A\,a^I_B\right)$	$X^A{}_B = \frac12\Big(\tilde a^A_I\,a^I_B - \tilde a^I_B\,a^A_I - a^A_I\,a^I_B + \tilde a^A_I\,\tilde a^I_B\Big)$	$X^a{}_b = \tilde a^a_i\,a^i_b - \tilde c^{\mathtt{i}}_b\,c^a_{\mathtt{i}} + \frac{p-q}{2}\,\delta^a_b, \qquad X^a{}_{\mathtt{b}} = -\tilde a^a_i\,\tilde b^i_{\mathtt{b}} + c^a_{\mathtt{i}}\,d^{\mathtt{i}}_{\mathtt{b}}$ $X^{\mathtt{a}}{}_{\mathtt{b}} = -\tilde b^i_{\mathtt{b}}\,b^{\mathtt{a}}_i + \tilde d^{\mathtt{a}}_{\mathtt{i}}\,d^{\mathtt{i}}_{\mathtt{b}} - \frac{p-q}{2}\,\delta^{\mathtt{a}}_{\mathtt{b}}, \qquad X^{\mathtt{a}}{}_b = a^i_b\,b^{\mathtt{a}}_i - \tilde c^{\mathtt{i}}_b\,\tilde d^{\mathtt{a}}_{\mathtt{i}}$
$\mathfrak{g}'$	$R_I{}^J = \frac12\left(\tilde a^A_I\,a^J_A - \tilde a^A_J\,a^I_A + \tilde a^A_I\,\tilde a^A_J - a^I_A\,a^J_A\right)$	$R_{\pm I}{}^J = R_I{}^J \pm (R_I{}^J)^*$ $R_I{}^J = \frac12\left(\tilde a^A_I\,a^J_A - \tilde b^A_J\,b^I_A + \tilde a^A_I\,\tilde b^A_J - b^I_A\,a^J_A\right)$	$R_I{}^J = \frac12\Big(\tilde a^A_I\,a^J_A - \tilde a^J_A\,a^A_I - a^A_I\,a^J_A + \tilde a^A_I\,\tilde a^J_A\Big)$	$R_j{}^i = \tilde a^a_j\,a^i_a - \tilde b^i_{\mathtt{a}}\,b^{\mathtt{a}}_j + \frac{m-n}{2}\,\delta^i_j, \qquad R_{\mathtt{j}}{}^i = a^i_a\,c^a_{\mathtt{j}} - \tilde b^i_{\mathtt{a}}\,\tilde d^{\mathtt{a}}_{\mathtt{j}},$ $R_{\mathtt{j}}{}^{\mathtt{i}} = -\tilde c^{\mathtt{i}}_a\,c^a_{\mathtt{j}} + \tilde d^{\mathtt{a}}_{\mathtt{j}}\,d^{\mathtt{i}}_{\mathtt{a}} - \frac{m-n}{2}\,\delta^{\mathtt{i}}_{\mathtt{j}}, \qquad R_j{}^{\mathtt{i}} = -\tilde a^a_j\,\tilde c^{\mathtt{i}}_a + b^{\mathtt{a}}_j\,d^{\mathtt{i}}_{\mathtt{a}}$

	$\big(O(p,q),Sp(2n,\mathbb{R})\big) \subset Sp(2n(p+q),\mathbb{R})$	$\big(O(N,\mathbb{C}),Sp(2M,\mathbb{C})\big) \subset Sp(4MN,\mathbb{R})$	$\big(O^*(2n),Sp(p,q)\big) \subset Sp(4n(p+q),\mathbb{R})$
$\dagger$	$(y^I_A)^\dagger = J_{IJ}\,y^J_A$ $A=(a,\mathtt{a}),\ a=1,\dots,p,\ \mathtt{a}=1,\dots,q$ $I=(+i,-i),\ i=1,\dots,n$	$(y^I_A)^\dagger = J_{IJ}\,y^{J*}_A$ $A=1,\dots,N, \quad I=(i,\mathtt{i}),\ i=1,\dots,M$	$(y^I_A)^\dagger = \Omega^{AB}\,\Psi_{IJ}\,y^J_B$ $A=(+a,-a), \quad a=1,\dots,n,$ $I=(+i,-i),\ i=(r,\mathtt{r}),\ r=1,\dots,p,\ \mathtt{r}=1,\dots,q$
$(\alpha,\tilde\alpha)$	$a^i_a = y^{+i}_a, \quad b^i_{\mathtt{a}} = y^{-i}_{\mathtt{a}}$	$a^I_A := \frac{1}{\sqrt2}\left(y^I_A + \eta_{IJ}\,y^{J*}_A\right)$	$a^a_R = \Omega_{RS}\,y^S_{+a}, \qquad b^a_{\mathtt{R}} = \Omega_{\mathtt{RS}}\,y^{\mathtt{S}}_{+a}$
$\mathfrak{g}$	$M_{ab} = \tilde a^a_i\,a^i_b - \tilde a^b_i\,a^i_a, \qquad M_{a\mathtt{b}} = \tilde a^a_i\,\tilde b^{\mathtt{b}}_i - a^i_a\,b^i_{\mathtt{b}}$ $M_{\mathtt{ab}} = \tilde b^{\mathtt{b}}_i\,b^i_{\mathtt{a}} - \tilde b^{\mathtt{a}}_i\,b^i_{\mathtt{b}}$	$M^\pm_{AB} := M_{AB} \pm (M_{AB})^*$ $M_{AB} = \frac12\left(\tilde a^A_I\,a^I_B - \tilde a^B_I\,a^I_A - \Omega^{IJ}\,\tilde a^{[A}_I\,\tilde a^{B]}_J + \Omega_{IJ}\,a^I_{[A}\,a^J_{B]}\right)$	$M_{+a\,-b} = -\big(\tilde a^R_b\,a^a_R - \tilde b^{\mathtt{R}}_a\,b^b_{\mathtt{R}} + (p-q)\,\delta_{ab}\big)$ $M_{+a\,+b} = a^{R[a}\,a^{b]}_R + \tilde b^{\mathtt{R}}_{[a}\,\tilde b_{b]\mathtt{R}}, \qquad M_{-a\,-b} = \tilde a_{R[a}\,\tilde a^R_{b]} + b^{[a}_{\mathtt{R}}\,b^{b]\mathtt{R}}$
$\mathfrak{g}'$	$K^{+i\,+j} = a^i_a\,a^j_a - \tilde b^{\mathtt{a}}_i\,\tilde b^{\mathtt{a}}_j, \qquad K^{-i\,-j} = \tilde a^a_i\,\tilde a^a_j - b^i_{\mathtt{a}}\,b^j_{\mathtt{a}},$ $K^{+i\,-j} = \tilde a^a_j\,a^i_a - \tilde b^{\mathtt{a}}_i\,b^j_{\mathtt{a}} + \frac{p-q}{2}\,\delta^i_j$	$K^{IJ}_\pm := K^{IJ} \pm (K^{IJ})^*,$ $K^{IJ} = \frac12\left(a^I_A\,a^J_A + \Omega^{IK}\,\Omega^{JL}\,\tilde a^A_K\,\tilde a^A_L + 2\,\Omega^{K(I}\,\tilde a^A_K\,a^{J)}_A\right)$	$K^{RS} = 2\,\tilde a^{(R}_a\,a^{S)a}, \qquad K^{\mathtt{RS}} = 2\,\tilde b^{(\mathtt{R}}_a\,b^{\mathtt{S})a},$ $K^{R\mathtt{R}} = a^{Ra}\,b^{\mathtt{S}a} + \tilde a^R_a\,\tilde b^{\mathtt{R}}_a$

Other realizations

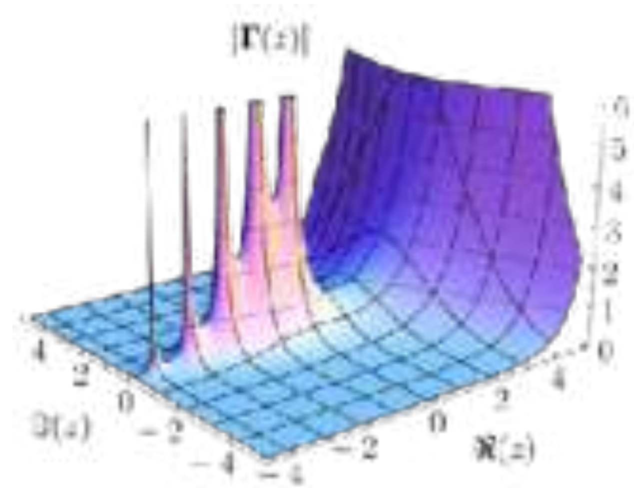
- Schrödinger realization

$$\hat{x}_i = \frac{1}{\sqrt{2}}\left(a_i + a_i^\dagger\right), \qquad \hat{p}_i = \frac{i}{\sqrt{2}}\left(a_i^\dagger - a_i\right).$$



- Bargmann-Segal realization

$$a_i^\dagger = z_i\,, \qquad a_i = \frac{\partial}{\partial z_i}\,.$$



Seesaw Pairs

- Dual pairs (G, G') and $(\tilde{G}, \tilde{G}')$ in $Sp(2N, \mathbf{R})$

$$\begin{array}{ccc} G & & \tilde{G}' \\ \cup & \times & \cup \\ \tilde{G} & & G' \end{array}$$

- $\mathrm{Hom}_{\tilde{G}}(\pi_{\tilde{G}}, \pi_G|_{\tilde{G}}) \cong \mathrm{Hom}_{G'}(\pi_{G'}, \pi_{\tilde{G}'}|_{G'})$

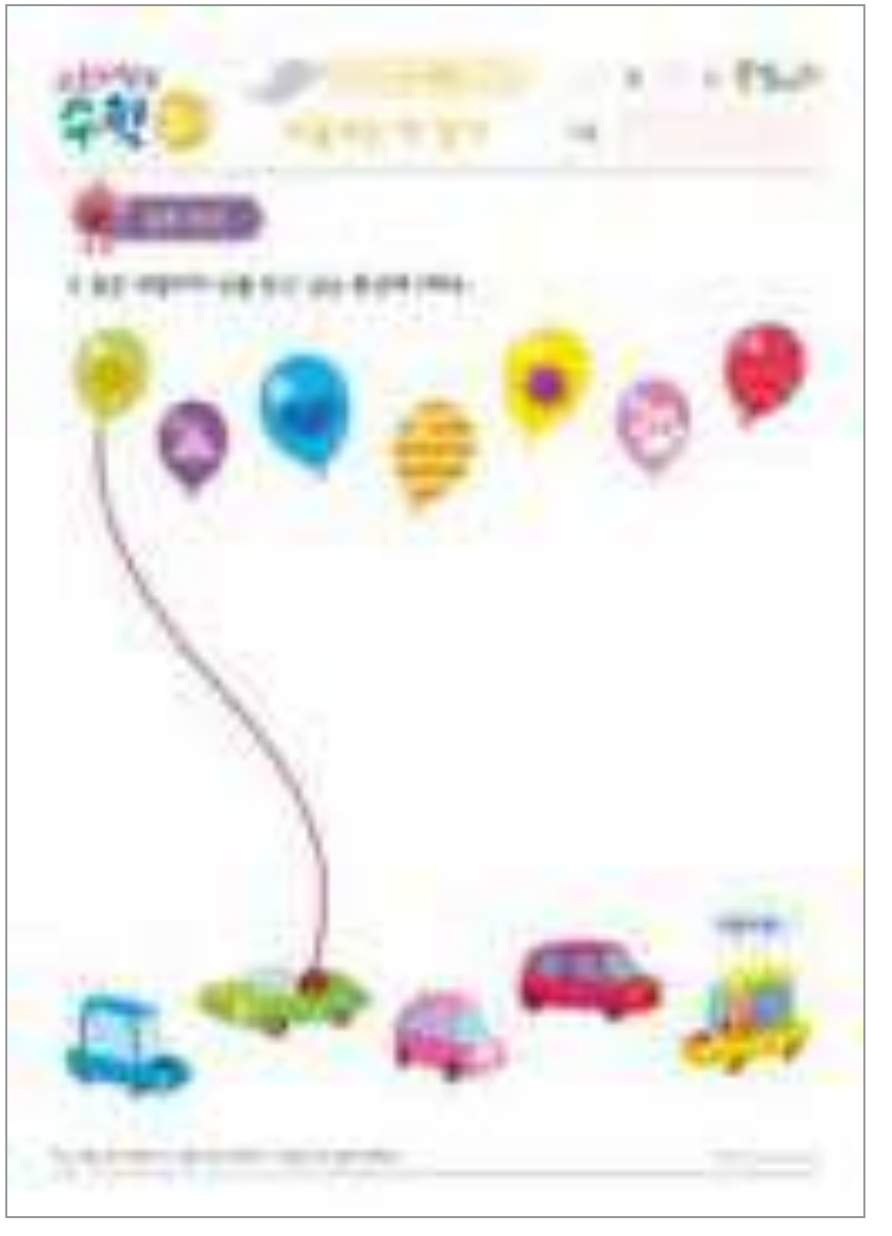
$$\mathrm{mult}_{\pi_G}(\pi_{\tilde{G}}) = \mathrm{mult}_{\pi_{\tilde{G}'}}(\pi_{G'})$$

$$\begin{array}{ccc} \pi_G & & \pi_{\tilde{G}'} \\ \downarrow & \times & \downarrow \\ \pi_{\tilde{G}} & & \pi_{G'} \end{array}$$

- Useful in deriving the correspondences

Correspondences

- Compact dual pairs
- “Exceptionally compact” dual pairs
- Simplest non-compact dual pairs
- More general non-compact dual pairs



Examples

1. ($U(M)$, $U(N)$)

2. ($U(M+,M-)$, $U(N)$)

    - AdS_5/CFT_4 : ($U(2,2)$, $U(N)$)

3. ($GL(1,\mathbf{C})$, $GL(N,\mathbf{C})$)

4. ($GL(M,\mathbf{C})$, $GL(N,\mathbf{C})$)

    - 4d Scattering Amplitude : ($GL(2,\mathbf{C})$, $GL(N,\mathbf{C})$)

Ex1. ($U(M)$, $U(N)$)

$$X^a{}_b = \tilde{a}^a_i \, a^i_b + \frac{N}{2} \, \delta^a_b \, , \qquad R_j{}^i = \tilde{a}^a_j \, a^i_a + \frac{M}{2} \, \delta^i_j$$

Ex2. (*U*(*M*+,*M*-) , *U*(*N*))

$$X^a{}_b = \tilde{a}^a_i\, a^i_b + \tfrac{N}{2}\, \delta^a_b\,, \quad X^{\mathtt{a}}{}_{\mathtt{b}} = -\tilde{b}^i_{\mathtt{b}}\, b^{\mathtt{a}}_i - \tfrac{N}{2}\, \delta^{\mathtt{a}}_{\mathtt{b}}\,, \quad X^a{}_{\mathtt{b}} = -\tilde{a}^a_i\, \tilde{b}^i_{\mathtt{b}}\,, \quad X^{\mathtt{a}}{}_b = a^i_b\, b^{\mathtt{a}}_i$$

$$R_j{}^i = \tilde{a}^a_j\, a^i_a - \tilde{b}^i_{\mathtt{a}}\, b^{\mathtt{a}}_j + \tfrac{M_+ - M_-}{2}\, \delta^i_j\,,$$

$$\begin{array}{ccc} U(M_+, M_-) & & U(N)\times U(N) \\ \cup & \times & \cup \\ U(M_+)\times U(M_-) & & U(N) \end{array}$$

Ex2. ($U(M+,M-)$, $U(N)$)

- $M_+ = M_- = 2$: $SU(2,2) \cong SO(2,4)$

$$[(s_1+s_2+n,n)\oslash(s_1-s_2+k,k)]_{U(N)}$$
$$\updownarrow$$
$$[2(s_2+n-k)]_{U(1)}\otimes\mathcal{D}_{\widetilde{SO}^+(2,4)}\big(s_1+n+k+N;s_1,s_2\big)$$

- N=1 : $$[\pm 2s]_{U(1)} \quad\longleftrightarrow\quad [\pm 2s]_{U(1)}\otimes\mathcal{D}_{\widetilde{SO}^+(2,4)}\big(s+1;s,\pm s\big)$$
- N=2 : $$[(s_1+s_2,s_2-s_1)]_{U(2)} \quad\longleftrightarrow\quad [2s_2]_{U(1)}\otimes\mathcal{D}_{\widetilde{SO}^+(2,4)}(s_1+2;s_1,s_2)$$

$$[(2s+n,n)]_{U(2)} \quad\leftrightarrow\quad [2(s+n)]_{U(1)}\otimes\mathcal{D}_{\widetilde{SO}^+(2,4)}\big(s+n+2;s,s\big)\,,$$
$$[(-n,-2s-n)]_{U(2)} \quad\leftrightarrow\quad [-2(s+n)]_{U(1)}\otimes\mathcal{D}_{\widetilde{SO}^+(2,4)}\big(s+n+2;s,-s\big)\,.$$

- N=3 : $$[(s_1+s_2+n,n,s_2-s_1)]_{U(3)} \quad\leftrightarrow\quad [2(s_2+n)]_{U(1)}\otimes\mathcal{D}_{\widetilde{SO}^+(2,4)}\big(s_1+n+3;s_1,s_2\big)$$
$$[(s_1+s_2,-n,s_2-s_1-n)]_{U(3)} \quad\leftrightarrow\quad [2(s_2-n)]_{U(1)}\otimes\mathcal{D}_{\widetilde{SO}^+(2,4)}\big(s_1+n+3;s_1,s_2\big)$$

Ex3. (*GL*(1,C) , *GL*(*N*,C))

$$X^A{}_B = \tfrac{1}{2}\left(\tilde{a}^A\, a_B - \tilde{b}_B\, b^A + \tilde{a}^A\, \tilde{b}_B - b^A\, a_B\right), \qquad Z_+ = \tilde{a}^A\, \tilde{b}_A - a_A\, b^A\,, \qquad Z_- = \tilde{a}^A\, a_A - \tilde{b}_A\, b^A$$

$$GL(N,\mathbb{C}) \qquad\qquad U(1,1) \cong SL(2,\mathbb{R}) \rtimes U(1)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$U(N) \qquad\qquad GL(1,\mathbb{C}) \cong \mathbb{C}^\times \cong \mathbb{R}^+ \times U(1)$$

$$H = \tilde{a}^A\, a_A + \tilde{b}_A\, b^A + N$$

$$E = \tilde{a}^A\, \tilde{b}_A\,, \quad F = -a_A\, b^A$$

$$J = \tilde{a}^A\, a_A - \tilde{b}_A\, b^A$$

$$Z_+ = E + F\,, \qquad Z_- = J$$

Ex3. (*GL*(1,C) , *GL*(*N*,C))

- Schrödinger realization $\omega_A = \frac{\partial}{\partial z^A}\,, \quad {\omega_A}^* = \frac{\partial}{\partial \bar{z}^A}\,, \quad \tilde{\omega}^A = z^A\,, \quad \tilde{\omega}^{A*} = \bar{z}^A$

$$X^A{}_B = z^A \frac{\partial}{\partial z_B} + \frac{1}{2}\,\delta^A_B\,, \qquad Z_+ = z^A \frac{\partial}{\partial z^A} + \bar{z}^A \frac{\partial}{\partial \bar{z}^A} + N$$

$$Z_- = z^A \frac{\partial}{\partial z^A} - \bar{z}^A \frac{\partial}{\partial \bar{z}^A}$$

- Determinant homogeneity condition ➔ **C**P$_N$

$$\langle z\,|\,U_{\mathcal{W}}(g)\,|\Psi_{\zeta,m}\rangle = |\det g|\,\langle z\,g|\Psi_{\zeta,\pm}\rangle \qquad [\,g \in GL(N,\mathbb{C})\,]\,,$$

$$\langle z\,|\,U_{\mathcal{W}}(a)\,|\Psi_{\zeta,m}\rangle = |a|\,\langle a\,z\,|\Psi_{\zeta,\pm}\rangle \qquad [\,a \in GL(1,\mathbb{C})\,]\,,$$

Ex4. ($GL(M,\mathbf{C})$, $GL(N,\mathbf{C})$)

- Schrödinger realization
- Determinant homogeneity condition
  (the most degenerate principal series representation)

  ➔ Complex Grassmannian $Gr_{M,N}(\mathbf{C})$

$$\langle z|\, U_{\mathcal{W}}(g)\, |\Psi_{\zeta,m}\rangle = |\det g|^{M} \, \langle z\, g\, |\Psi_{\zeta,m}\rangle \qquad [\, g \in GL(N,\mathbb{C})\,]$$

$$\begin{aligned} \langle z|\, U_{\mathcal{W}}(h)\, |\Psi_{\zeta,m}\rangle &= |\det h|^{N} \, \langle h^{t}\, z\, |\Psi_{\zeta,m}\rangle \\ &= |\det h|^{i\,\zeta} \left(\frac{\det h}{|\det h|} \right)^{m} \langle z\, |\Psi_{\zeta,m}\rangle \qquad [\, h \in GL(M,\mathbb{C})\,] \end{aligned}$$

Ex4. (*GL*(*M*,**C**) , *GL*(*N*,**C**))

- *M*=2 : Scattering amplitudes of 4d CFT fields

$$
\begin{array}{ccc}
U(2,2) & & GL(N_+ + N_-, \mathbb{C}) \\
\cup & \times & \cup \\
GL(2,\mathbb{C}) & & U(N_+, N_-)
\end{array}
$$

3d Parallelism:
($Sp(2M,\mathbf{R})$, $O(N)$) and ($GL(M,\mathbf{R})$, $GL(N,\mathbf{R})$)

- **Correspondence**
  - ($Sp(2M,\mathbf{R})$, $O(N)$)
  - ($GL(M,\mathbf{R})$, $GL(N,\mathbf{R})$)
- **Physical Application**
  - AdS_4/CFT_3 : ($Sp(4,\mathbf{R})$, $O(N)$)
  - 3d Scattering Amplitude : ($GL(2,\mathbf{R})$, $GL(N,\mathbf{R})$)

6d Parallelism: $(O^*(2M) , Sp(N))$ and $(GL(M,\mathbf{H}) , GL(N,\mathbf{H}))$

- **Correspondence**
  - $(O^*(2M) , Sp(N))$
  - $(GL(M,\mathbf{H}) , GL(N,\mathbf{H}))$
- **Physical Application**
  - AdS_7/CFT_6 : $(O^*(8) , Sp(N))$
  - 6d Scattering Amplitude : $(GL(2,\mathbf{H}) , GL(N,\mathbf{H}))$

dS representations

- dS3: (*Sp*(2,**C**) , *O*(1,**C**))

$$[(\bar{n})]_{O(1,\mathbb{C})} = [(-1)^{\bar{n}}]_{\mathbb{Z}_2} \quad \longleftrightarrow \quad \begin{array}{l} \pi_{Sp(2,\mathbb{C})}(0) \;\; : \;\; (\mu, j) = (i\,\frac{1}{2}\,, 0) \\ \pi_{Sp(2,\mathbb{C})}(1) \;\; : \;\; (\mu, j) = (0\,, \frac{1}{2}) \end{array}$$

- dS4: (*Sp*(1,1) , *O**(2))

$$[\pm 2s]_{O^*(2)} \quad \longleftrightarrow \quad \pi_{\widetilde{SO}^+(1,4)}\big([(s, \pm s)]_{\widetilde{SO}(4)}\big)$$

- dS5: (*U**(4) , *U**(2))

$$[\zeta]_{\mathbb{R}^+} \otimes [m]_{SU(2)} \quad \longleftrightarrow \quad [\zeta]_{\mathbb{R}^+} \otimes \pi_{\widetilde{SO}^+(1,5)}(\zeta, m)$$

Branching properties

- **Singleton**, conformal field representations of *SO*(2,*d*)
  - **Single sum** decomposition under *SO*(2)x*SO*(*d*) restriction
  - (Almost) **irreducibility** under restriction to isometry groups

$$
\begin{array}{ccccc}
\pi_G(\zeta) & G & & \tilde{G}' & \pi_{\tilde{G}'}(\theta(\tilde{\zeta})) \\
\downarrow & \cup & \nwarrow\!\!\nearrow \; \swarrow\!\!\searrow & \cup & \downarrow \\
\pi_{\tilde{G}}(\tilde{\zeta}) & \tilde{G} & & G' & \pi_{G'}(\theta(\zeta))
\end{array}
$$

Branching properties

- **Ex1**: Restriction of 4d conformal fields to **AdS4**

$$\begin{array}{ccc} \mathcal{D}_{U(N,N)}\big([(\tfrac{|n|+n}{2}),\tfrac{1}{2}]_{U(N)} \otimes [(\tfrac{|n|-n}{2}),\tfrac{1}{2}]_{U(N)}\big) & & [(|n|)]_{O(2)} \\ \downarrow & \times & \downarrow \\ \mathcal{D}_{Sp(2N,\mathbb{R})}\big([(|n|),1]_{U(N)}\big) & & [(n)]_{U(1)} \end{array}$$

- **Ex2**: Restriction of 4d conformal fields to **dS4**

$$\begin{array}{ccc} \mathcal{D}_{U(2N_+,2N_-)}\big(\big[\big(\tfrac{|n|+n}{2}\big),\tfrac{1}{2}\big]_{U(2N_+)} \otimes \big[\big(\tfrac{|n|-n}{2}\big),\tfrac{1}{2}\big]_{U(2N_-)}\big) & & [n+N_+-N_-]_{O^*(2)} \\ \downarrow & \times & \downarrow \\ \pi_{Sp(N_+,N_-)}\big(\big[\big(\tfrac{|n|+n}{2}\big)\big]_{Sp(N_+)} \otimes \big[\big(\tfrac{|n|-n}{2}\big)\big]_{Sp(N_-)}\big) & & [n+N_+-N_-]_{U(1)} \end{array}$$

Casimirs

- (GL_M , GL_N) $$x(t) = \sum_{n=0}^{\infty} t^n\, \mathcal{C}_n[\boldsymbol{X}]\,, \qquad r(t) = \sum_{n=0}^{\infty} t^n\, \mathcal{C}_n[\boldsymbol{R}]$$

$$x(t) = \frac{1}{1-\frac{N}{2}\,t}\left[\frac{1+\frac{N-2\,M}{2}\,t}{1+\frac{N-M}{2}\,t}\;r\left(\frac{t}{1+\frac{N-M}{2}\,t}\right) + M - N\right]$$

- (O_N , Sp_{2M}) $$m(t) = \sum_{n=0}^{\infty} t^n\, \mathcal{C}_n[\boldsymbol{M}]\,, \qquad k(t) = \sum_{n=0}^{\infty} t^n\, \mathcal{C}_n[\boldsymbol{K}]$$

$$k(t) = \frac{(2M-N)\left(1+\frac{2Mt}{2+Nt}\right) + m\left(\frac{2t}{2+(2M+N)t}\right)}{(1+\frac{N}{2}t)\left(1+\frac{2Mt}{2+Nt}\right) - \frac{1}{2}t\left(1+\frac{1+Mt}{1+(1+M)t}\right) m\left(\frac{2t}{2+(2M+N)t}\right)}$$

Conclusion

- **What I didn't talk about**
  - Role of discrete subgroups
  - Plethysms
- **Future directions**
  - Correspondences in more general dual pairs
  - Exploration of physical applications
  - Fermionic and supersymmetric cases
  - Other embedding groups

Thank you (and have a nice vacation)